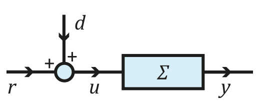
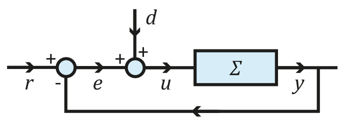
Control Systems I — *Max Schaldach*

1. Introduction

1.1. Systems

A **system** takes a controllable input u and translates it into an **output** y. It is thus nothing else than a function $y = \Sigma(u)$

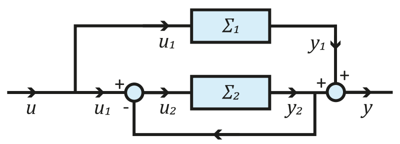
Open loop / feed-forward	Closed loop / feedback
	
Requires precise knowledge of the system & environment, as the output can't be corrected	Stabilizes unstable systems, but can also cause instability in an otherwise stable system

r is the **reference value** (the desired output), d are **disturbances** (also called exogenous input, e.g. wind), and e is the **negative error**:

$$negative\ error = reference\ value - output$$

1.2. Linear Systems

Linear systems satisfy $\Sigma(au_1 + bu_2) = a\Sigma(u_1) + b\Sigma(u_2)$ and $\Sigma(k \cdot u) = k \cdot \Sigma(u)$. These properties can be used to simplify:



$$y = \left(\Sigma_1 + \frac{\Sigma_2}{1+\Sigma_2}\right)(u)$$

Hint: Treat Σ as scalars: $\Sigma(u) = \Sigma \cdot u$

Linear: $y(t) = 3u(t^2)$, $y(t) = u(t) + u(t-1)$, $y(t) = \int_{-\infty}^{t} u(\tau)\, d\tau$

Non-linear: $y(t) = 1 + u(t)$, $y(t) = \int_0^t u^2(\tau)\, d\tau$, $y(t) = \Re(u(t))$

1.3. Static, Causal & Time-Invariant Systems

Static / memoryless: Output only depends on current input $u(t)$

Non-static: $y(t) = u(t-1)$, $y(t) = u(t) + x(t)$, $y(t) = \int_0^t u(\tau)\, d\tau$

Causal: Output relies on past and current, but not on future input. Non-causal systems *cannot* be implemented in the real world

Non-causal: $y(t) = u(t+2)$, $y(t) = \frac{d}{dt}u(t)$, $y(t) = \int_{-\infty}^{t+1} u(\tau)\, d\tau$

Time-invariant: Shifting output has same effect as shifting input

Time-*varying*: $y(t) = t \cdot u(t)$, $y(t) = u(\sin(t))$, $y(t^2) = u(t)$

2. State-Space Form

2.1. Introduction

LTI systems are dynamic, linear and time-invariant. They are fully described by two formulas, the so-called **state-space form**:

$$\begin{cases} \dot{x}(t) \coloneqq f(t) = A \cdot x(t) + B \cdot u(t) \\ y(t) \coloneqq g(t) = C \cdot x(t) + D \cdot u(t) \end{cases}$$

$x(t)$ is the system's **state**, a vector of variables representing the properties of the system at time t (e.g., angle of a pendulum). The size n of the vector $x(t)$ is called the **dimension** of the system. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}^{1 \times 1}$ are parameters

$$\begin{cases} x(t) = e^{At} \cdot x_0 + \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau \\ y(t) = C \cdot e^{At} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau + D \cdot u(t) \end{cases}$$

where $x_0 \coloneqq x(t=0)$ is the **initial state** of the system

The first term in y is the **initial condition response** y_{IC}, dominating the short-term. The *rest* is the **forced response** y_F, reflecting the long-term response. The third term is also called **feedthrough**

2.2. Interlude – Linear Algebra

The inverse of a 2×2 matrix A is computed as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

Adjoint 3x3:

$$adj\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

If A is in diagonal form, e^{At} can be simplified to:

$$\exp\left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t\right] = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

If A is not diagonal but diagonalizable, e^{At} is computed using:

$$\det\left(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}\right) = (a - \lambda)(d - \lambda) - bc = 0$$

$$e^{At} = \mathrm{diag}\left[e^{\lambda_1 t}, e^{\lambda_2 t}\right]$$

If A is in Jordan form, e^{At} can be simplified to:

$$\exp\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t\right) = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\exp\left(\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} t\right) = e^{\lambda t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

2.3. Stability from state-space form

The eigenvalues of A determine the **stability** of the system:

Asymptotically stable: State converges to "zero" (stays constant) for bounded initial condition & zero input. $\Re(\lambda_i) < 0$ for all λ_i of A

Lyapunov stable: State will remain bounded for bounded initial condition & zero input. $\Re(\lambda_i) \le 0$ for all λ_i of A

Unstable: State does not remain bounded. $\Re(\lambda_i) > 0$ for at least one λ_i of A

BIBO stable: Output remains bounded for *every* bounded input. This is satisfied if the system response to an impulse $y_\delta(t) \to 0$ and its integral settles to a finite value $\int_{-\infty}^{\infty} |y_\delta(t)| dt < \infty$

- Asymptotically stable → BIBO stable & Lyapunov stable
- Asymptotically stable ↔ BIBO stable (for **minimal** LTI systems)

The initial condition response y_{IC} depends on the eigenvalues of A:

There are no oscillations if $\Im(\lambda_i) = 0$. Oscillations neither affect the stability nor the over- and undershoot (see 3.2.)

2.4. Linearization

A system is at **equilibrium** when for a given input u_e, $x(t)$ remains constant over time $\dot{x}(t) = f(x_e, u_e) = 0$. Non-linear systems can be **linearly approximated** near their **equilibrium points** (x_e, u_e):

$$\begin{cases} \dot{x} = f(x(t), u(t)) \\ y = g(x(t), u(t)) \end{cases} \to \begin{cases} \dot{x} = A \cdot x + B \cdot x \\ y = C \cdot x + D \cdot x \end{cases}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x_e, u_e}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}_{x_e, u_e}$$

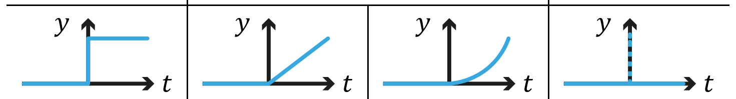
$$C = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}_{x_e, u_e}, \quad D = \begin{bmatrix} \frac{\partial g(x, u)}{\partial u} \end{bmatrix}_{x_e, u_e}$$

Hartman-Grobman theorem: if the linearization is asymptotically stable, then the non-linear system is locally asymptotically stable

3. Transfer Functions

3.0. Signals & Responses

The most commonly used reference signals are:



Step	Ramp		Impulse
$r(t) = h(t)$	$t \cdot h(t)$	$t^2/_2 \cdot h(t)$	$\delta(t)$
$R(s) = 1/_s$	$1/_{s^2}$	$1/_{s^3}$	1

where $R(s)$ is the Laplace transformation of $r(t)$

3.1. Introduction

The state-space form is tedious to solve. The Laplace transformation of $y(t)$ simplifies the calculations significantly:

$$Y(s) = (C \cdot (s\mathbb{I} - A)^{-1} \cdot B + D) \cdot U(s)$$

Caution: This formula only holds true if s is not an eigenvalue of A

We define $G(s) = C \cdot (s \cdot \mathbb{I} - A)^{-1} \cdot B + D$ as the **transfer function** of a system as it relates the output Y to the input U:

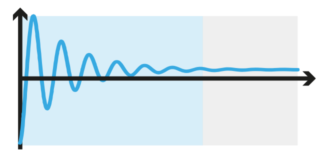
$$Y(s) = G(s) \cdot U(s)$$

$$Y_{ss} = G(s) \cdot e^{st}$$

The input $u = e^{st}, s \in \mathbb{C}$ can represent a variety of inputs (e.g. $s = 0$ is a constant input and $\Im(s) \neq 0$ is a sinusoidal input):

$$y(t) = Ce^{At}(x_0 - (s\mathbb{I} - A)^{-1}B) + (C(s\mathbb{I} - A)^{-1}B + D)e^{st}$$

Where the first term is the **transient response** and the second is the **steady-state response** y_{ss}. If the system is asymptotically stable, the transient response (blue) converges to zero and the system reaches a **steady-state** (grey):



Transfer functions are often presented in the **canonical form**:

$$G(s) = \frac{b_m \cdot s^m + b_{m-1} \cdot s^{m-1} + \cdots + b_0}{s^n + a_{n-1} \cdot s^{n-1} + \cdots + a_0} + d$$

$$A = \begin{bmatrix} 0 & 1 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ \vdots & \vdots & \ddots & & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \in n \times n, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in n \times 1$$

$$C = [b_0 \ \ \ldots \ \ b_m \ \ 0 \ \ \ldots \ \ 0] \in 1 \times n, \quad D = d$$

$G(s)$ is **proper** if $n \ge m$, which also means that the system is causal

3.2. Root Locus Form

Root locus is an alternative representation of transfer functions:

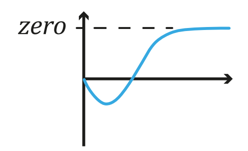
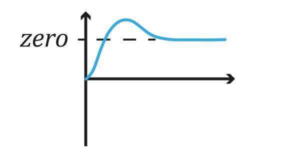
$$G(s) = k_{locus} \cdot \frac{(s - z_1) \cdot (s - z_2) \cdot \ldots \cdot (s - z_i)}{(s - p_1) \cdot (s - p_2) \cdot \ldots \cdot (s - p_j)}$$

where z_i are the roots of the numerator (**zeros**) and p_j are the roots of the denominator (**poles**). k_{locus} is a constant. The poles are part of the eigenvalues of A and therefore determine the system's stability:

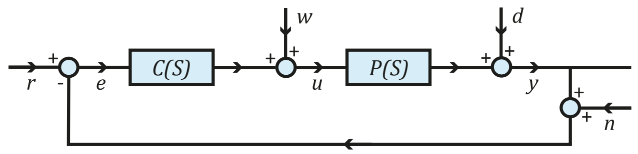
- A system is unstable if at least one pole has a positive real part
- The response oscillates if there is a pole with an imaginary part
- The further a pole from the origin, the faster the response converges to zero

Caution: Stability information is lost when an unstable pole is canceled with a zero. Transfer functions with no identical poles and zeros and no pole-zero cancellations are called **minimal**

The zeros influence the shape of the initial condition response y_{IC}:

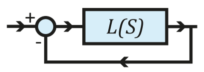
Non-minimum phase system	Minimum phase system
$\Re(z_i) > 0$ causes **undershoot**:	$\Re(z_i) < 0$ causes **overshoot**:
	
System initially reacts in the opposite direction of the steady-state value	System initially reacts by exceeding the steady-state value
The more positive $\Re(z_i)$, the smaller the undershoot	The more negative $\Re(z_i)$, the smaller the overshoot

3.3. Controllers

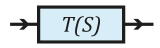


where C is the controller and P the plant / system. w is the input disturbance, d is the output disturbance, and n is sensor noise

$$Y(s) = S(s) \cdot (D(s) + P(s) \cdot W(s)) + T(s) \cdot (R(s) - N(s))$$

Open loop: 

$$L(s) = C(s) \cdot P(s)$$

Closed loop: 

$$T(s) = \frac{L(s)}{1 + L(s)}, \quad S(s) = \frac{1}{1 + L(s)}$$

Ohne W: R to Y = T(s), D to Y = S(s), N to Y = -T(s), R to E = S(s)

3.4. Root Locus Plot

An **absolute stable** system is stable for all values of k

Number of branches is equal to the denominator order of T

Asymptotes

Contact point / Centroid of asymptotes

$$S_{com} = \frac{\sum x_{Poles} - \sum x_{Zeros}}{\#Poles - \#Zeros}$$

$$x_i \to Coordinates\ on\ the\ Real\ axis$$

Angle of asymptotes

$$\alpha_n = \frac{2n+1}{\#Poles - \#Zeros} \cdot 180°$$

$$n = \{0; 1; \ldots; (\#Poles - \#Zeros - 1)\}$$

Für negative K wird der Plot um +180° erweitert, also wird die Formel für die Angles nur noch $= 2n \cdot 180°$

3.5. Partial Fraction Form

Partial fraction is another representation of transfer functions:

$$G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n} + r_0$$

where $r_0 = d$, p_i are the poles and r_i are the **residues**:

p_i is a single pole:	$r_i = \lim\limits_{s \to p_i}[(s - p_i) \cdot G(s)]$
p_i is repeated m times:	$r_i = \frac{1}{(m-1)!} \lim\limits_{s \to p_i}\left[\frac{d^{m-1}}{ds^{m-1}}((s - p_i)^m \cdot G(s))\right]$

The impulse response can be written in terms of the residues:

$$y_\delta(t) = r_1 \cdot e^{p_1 t} + r_2 \cdot e^{p_2 t} + \dots + r_n \cdot e^{p_n t}$$

3.5. Initial / Final Value Theorem & Steady-State Error

The long- and short-term output of a system are calculated using:

Initial value theorem:	$\lim_{t\to 0}[y(t)] = \lim_{s\to\infty}[s \cdot Y(s)]$
Final value theorem:	$\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}[s \cdot Y(s)]$

The latter only applies to stable systems with at most one integrator

The **steady-state error** e_{ss} of the *closed*-loop system is the difference between the reference value and the output for $t \to \infty$:

$$e_{ss} = \lim_{t\to\infty}[r(t) - y(t)] = \lim_{s\to 0}\left[\frac{s \cdot R(s)}{1 + L(s)}\right]$$

Alternatively, e_{ss} can be derived from the number of poles of $L(s)$ at the origin (so-called **integrators**) and the reference signal $r(t)$:

	$r(t) = h(t)$	$r(t) = t \cdot h(t)$	$r(t) = \frac{t^2}{2} \cdot h(t)$
0 integrators	${}^1/_{1 + k_{bode}}$	∞	∞
1 integrator	0	${}^1/_{k_{bode}}$	∞
2 integrators	0	0	${}^1/_{k_{bode}}$

where $k_{bode} = \lim_{s\to 0}[L(s)]$

3.6. Dominant Poles Approximation

A system of higher order can be approximated by a system of lower order by only keeping the **dominant poles**. Dominant poles are those closest to 0 and who are far away from the zeros

- If $p_i \approx z_i$ with $\Re(p_i) < 0$ and $\Re(z_i) < 0$, cancel p_i and z_i
- Cancel poles with $\Re(p_i) \ll 0$ relative to other poles

Caution: Add a gain k to $G_{approx}(s)$ so that $G_{approx}(0) = G(0)$

$$G(s) = \frac{s + 0.6}{(s + 1 + 5j)(s + 1 - 5j)(s + 3)(s + 0.5)}$$

Cancel $p_1 = s + 0.6$ and $z_4 = s + 0.5$ as lie close to each other:

$$G_{approx}(s) \cong \frac{1}{(s + 1 + 5j)(s + 1 - 5j)(s + 3)}$$

Cancel $z_3 = s + 3$ as it has the most negative real part:

$$G_{approx}(s) \cong \frac{1}{(s + 1 + 5j)(s + 1 - 5j)}$$

Add a gain so that $G_{approx}(0) = G(0)$:

$$G_{approx}(s) = \frac{2}{5} \cdot \frac{1}{(s + 1 + 5j)(s + 1 - 5j)}$$

Hint: k is equal to the ratio of the cancelled zeros and poles

3.7. System Characteristics

When designing a system, it can be engineered to have certain features. In the following, we will look at 2nd order systems of the form $G(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta \cdot \omega_n \cdot s + {\omega_n}^2}$ with the poles $p_{1,2} = \sigma \pm j\omega$:

Settling time: Time until the response is within $\pm d\%$ (often 2%) of the steady-state:

$$T_d = -\frac{1}{|\sigma|} \ln\left(\frac{d}{100}\right)$$

where $|\sigma| = \zeta\omega_n$ and $\omega_n = \sqrt{\sigma^2 + \omega^2}$

Time to peak: Time until the highest value of the response is reached:

$$T_p = \frac{\pi}{\omega}$$

Rise time: Time until the reference value is reached for the first time:

$$T_{100\%} = \frac{\frac{\pi}{2} - \varphi}{\omega} \approx \frac{\pi}{2\omega_n}$$

$$T_{90\%} = (0.14 + 0.4\zeta) \cdot \frac{2\pi}{\omega_n}$$

where $\varphi = \arctan\left(\frac{\sigma}{\omega}\right)$ and $|\varphi| = \arcsin(\zeta)$

The more negative the real part, the smaller $T_{100\%}$

Peak overshoot: The maximum overshoot compared to the reference value:

$$M_p = e^{\frac{\sigma \cdot \pi}{\omega}}$$

where $\zeta^2 = \frac{\ln(M_p)^2}{\pi^2 + \ln(M_p)^2}$

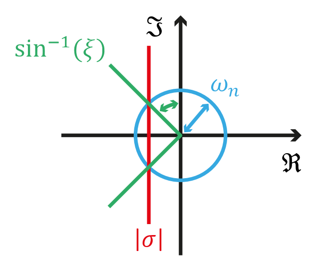
The larger the imaginary part, the larger M_p

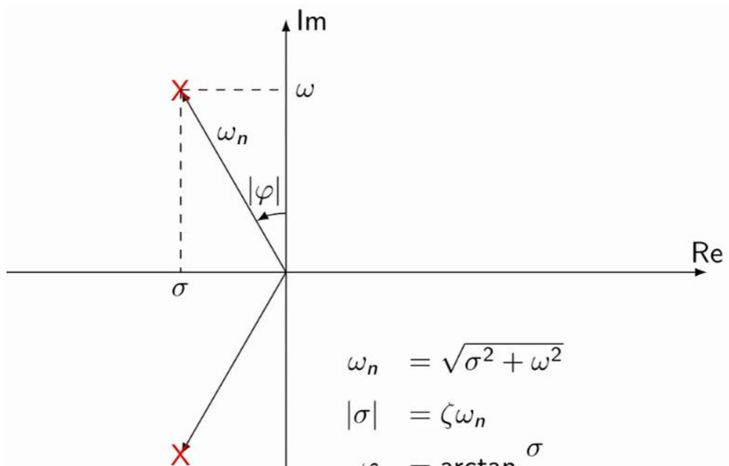
The above features yield two formulas for calculating ζ and ω_n:

$$\zeta^2 = \frac{\ln(M_p)^2}{\pi^2 + \ln(M_p)^2}, \qquad \omega_n = (0.14 + 0.4\zeta) \cdot \frac{2\pi}{T_{90\%}}$$

The location of the poles can also be derived visually:

- Poles with short settling times lie left of the red line
- Poles between the green lines have small overshoots
- Poles outside the blue circle have small rise times





$$\omega_n = \sqrt{\sigma^2 + \omega^2}$$
$$|\sigma| = \zeta\omega_n$$
$$\varphi = \arctan\frac{\sigma}{\omega}$$
$$\zeta = \sin(|\varphi|)$$

4. PID Controllers

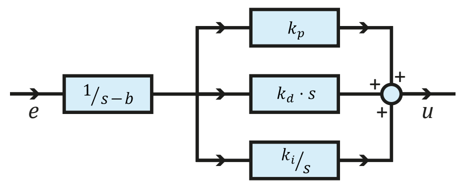
A **PID controller** creates an input signal to the plant that depends on the negative error e, the integral of e, and the derivative of e:

$$u(t) = k_p \cdot e(t) + k_i \cdot \int_0^t e(\tau)\,d\tau + k_d \cdot \frac{de(t)}{dt}$$

	Advantages	Disadvantages
Increase k_p	• e_{ss} decreases • Faster response • Increases bandwidth	• More sensitive to noise • More oscillations • Phase margin decreases (see 4.5.)
Increase k_d	• Less oscillations • Phase margin increases (see 4.5.) • Reduces overshoot	• More sensitive to noise • Slower response
Increase k_i	• e_{ss} decreases (to 0) • Faster response	• More oscillations • Phase margin decreases (see 4.5.)

$$C_{PID}(s) = k_d \cdot s + k_p + \frac{k_i}{s} = \frac{k_d \cdot s^2 + k_p \cdot s + k_i}{s}$$

With less poles than zeros, PID controllers are *non*-causal and hence not realizable. Therefore, an additional element with $C(s) = {}^1/_{s - b}$ is added to the system to make it causal:



Choose $\Re(b) \ll 0$ so that system characteristics remain unchanged

4. Bode & Nyquist Plots

$$\angle(a + jb) = \arctan\left(\frac{b}{a}\right) + \begin{cases} \pi, & a < 0,\ b \geq 0 \\ -\pi, & a < 0,\ b < 0 \\ 0 & \end{cases}$$

$$\arctan(-x) = -\arctan(x)$$

Decibels $X_{dB} = 20 \cdot \log_{10}(x)$ are converted to decimals as follows:

Decimal	Decibel	Decimal	Decibel
100	40	0.1	-20

4.1. Magnitude & Phase Shift

A sinusoidal input to an asymptotically stable system causes a sinusoidal steady-state response with the same frequency but a different **amplitude** and with a **phase shift**:

$$y_{ss}(t) = |G(j\omega)| \cdot \sin(t + \angle G(j\omega))$$

where ω is the **input frequency**, $|G(j\omega)|$ the amplitude (also M), and $\angle G(j\omega)$ the phase shift (also φ):

$$|G(j\omega)| = |k| \cdot \frac{\prod_i |j\omega - z_i|}{\prod_j |j\omega - p_j|}, \quad \angle G(j\omega) = \angle k \cdot \frac{\prod_i \angle(j\omega - z_i)}{\prod_j \angle(j\omega - p_j)}$$

Caution: Do not forget the factor k

4.2. Bode Form

Bode form is another way to represent transfer functions:

Same Bode Plot for 2 functions => Same function

$$G(s) = k_{bode} \cdot \frac{\left(\frac{s}{-z_1} + 1\right) \cdot \left(\frac{s}{-z_2} + 1\right) \cdot \ldots \cdot \left(\frac{s}{-z_i} + 1\right)}{\left(\frac{s}{-p_1} + 1\right) \cdot \left(\frac{s}{-p_2} + 1\right) \cdot \ldots \cdot \left(\frac{s}{-p_j} + 1\right)}$$

where $k_{bode} = {}^{b_0}/_{a_0} + d$ if derived from canonical form (see 3.1.)

4.3. Bode Plots

Nimm einfach die regeln und superponiere. Aber theoretisch ist der Plot für eine Zero die einem Pole entspricht, einfach der Plot des Poles gespiegelt, weil $Z = P^{-1}$

Phase \ Magnitude	-20 dB/dec	$+20$ dB/dec
$-90°$	stable pole	non-minimum phase zero
$+90°$	unstable pole	minimum phase zero

Differentiator "s" +20db/dec (bei Frequenz 0 = 0) Phase 90

Integrator "1/s" -20dB/dec (bei Frequenz 0 = 0) Phase -90

4.4. Nyquist Plots

Nyquist plots are another way to visualize how the magnitude $|G(j\omega)|$ and the phase $\angle G(j\omega)$ change with the frequency ω:

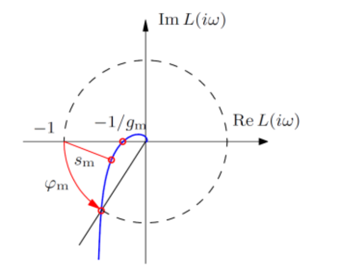
$$RHP\ poles\ of\ T(s) = \circlearrowright\ of\ -{}^1/_k +\ RHP\ poles\ of\ L(s)$$

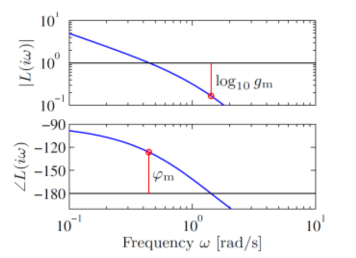
$$=>\ Z = N + P$$

4.5. Phase & Gain Margins

The **phase margin** PM / φ_m indicates how much the phase can be changed at the ω where the magnitude is 1 before encircling -1 (making the closed-loop system unstable). The **gain margin** GM / g_m indicates how much the magnitude can be changed at the ω where the phase is $\pm 180°$ before encircling -1

Phase margin:	Gain margin:
Find **cross-over frequency** ω_c: $\lvert L(j\omega_c)\rvert = 1$ $\lvert L(j\omega_c)\rvert_{dB} = 0$ Find the phase $\varphi = \angle L(j\omega_c)$ and set the phase margin: $\varphi_m = \varphi + 180°$	Find the frequency ω at which $\angle L(j\omega) = -180°$. Then find $\lvert L(j\omega)\rvert_{dB}$ for the same ω Compute the gain margin: $g_{m,dB} = 0 - \lvert L(j\omega)\rvert_{dB}$ $g_m = 10^{\frac{g_{m,dB}}{20}}$





Stability is guaranteed for $g_{m,dB} > 0$ / $|L(j\omega_c)|_{dB} < 0$ and $\varphi_m > 0$

Caution: These criterions only hold true if $L(s)$ is stable. If $L(s)$ is unstable, it might in fact be desirable to encircle $-1/k$

5. Compensators

5.0. Introduction

Noise $N(s)$ is cancelled when $T(s) \approx 0$ and disturbances $D(s)$ when $S(s) \approx 0$. Noise has high frequencies ($> 100Hz$) while disturbances have at low frequencies ($< 10Hz$). Therefore, $S(s) \approx 0$ is desired for low ω and $T(s) \approx 0$ for high ω

Bandwidth BW is defined as the ω for which $|T(j\omega)| > 1/\sqrt{2}$ and $|T(j\omega)|_{dB} > -3.01$ respectively. The bandwidth is approximately the cross-over frequency of the *open*-loop system

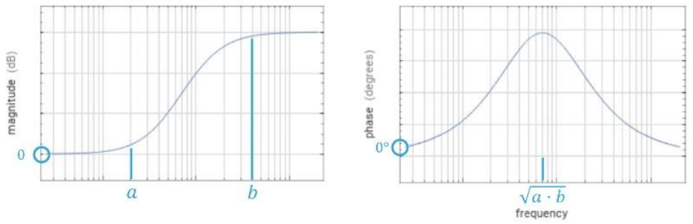
A higher gain and poles close to the origin increase the bandwidth. However, the stability might be lost, making the bandwidth irrelevant. Zeros close to the origin decrease the bandwidth

Hint: If the bandwidth is asked in Hz, use $f = {}^{\omega}/_{2\pi}$

5.1. Lead Compensators

A **lead compensator** is a loop element that increases the phase margin. However, it also makes a system more sensitive to noise:

$$C_{lead}(s) = \frac{s/a+1}{s/b+1} = \frac{b}{a}\cdot\frac{s+a}{s+b}, \qquad 0 < a < b$$



$|L(j\omega)|$ changes by $20dB$/decade. The maximum increase of φ is:

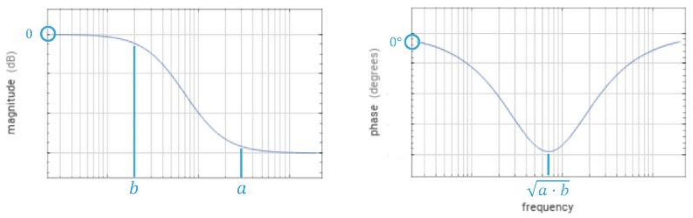
$$\Delta\varphi \approx 2\cdot\arctan\left(\sqrt{b/a}\right) - 90°$$

1. Pick the desired crossover frequency $\omega_c = \sqrt{a\cdot b}$
2. Pick b/a depending on the desired phase increase
3. Possibly add a proportional gain k to set ω_c back to the desired frequency ($|k\cdot L(j\omega)| = 1$)

5.2. Lag Compensators

A **lag compensator** is a loop element that improves disturbance rejection and command tracking by decreasing the sensitivity to noise. However, it also decreases the phase margin of the system:

$$C_{lag}(s) = \frac{s/a+1}{s/b+1} = \frac{b}{a}\cdot\frac{s+a}{s+b}, \qquad 0 < b < a$$



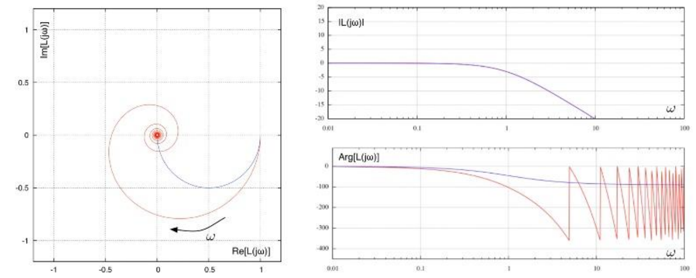
$|L(j\omega)|$ changes by $-20dB$/decade. The maximum decrease of φ is:

$$\Delta\varphi \approx 2\cdot\arctan\left(\sqrt{b/a}\right) - 90°$$

1. Choose a/b as the desired increase in magnitude at low ω
2. Pick $\sqrt{a\cdot b}$ as far as possible from the desired crossover frequency ω_c to not risk instability

6. Time-Delays

A **time-delay** element (linear) has the transfer function $C(s) = e^{-\tau\cdot s}$ where τ is the time-delay. It causes the phase to oscillate but doesn't change the magnitude of the open-loop system:



A time delay reduces the phase margin:

$$\varphi_{m,time\ delay} = \varphi_m - \omega_c\cdot\tau$$

where ω_c is the crossover frequency in rad and τ is in seconds

A time delay can be simplified using **Padé approximation**:

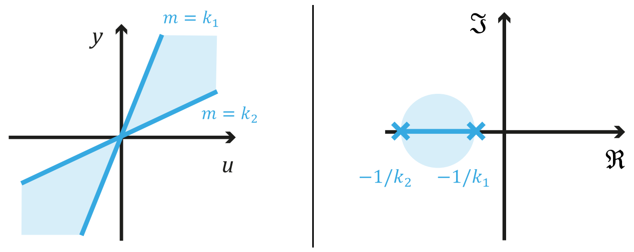
$$e^{-\tau\cdot s} \approx \frac{2/_{\tau} - s}{2/_{\tau} + s}$$

When increasing the time delay, the phase margin φ decreases. This means that the achievable bandwidth also decreases

7. Non-Linear Systems

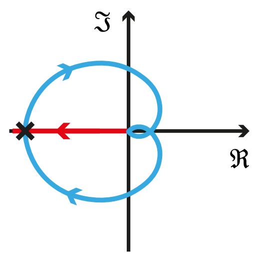
A system with a **non-linear** element is absolute stable, if, for any choice of the nonlinear element, the input $u = 0$ is a globally asymptotically stable equilibrium point of the closed-loop system

A system with a non-linear element that has a gain between k_1 and k_2 has as many unstable poles of $L(s)$ as *counter*clockwise encirclements of the circle drawn from $-1/k_1$ to $-1/k_2$



Encircling the line between $-1/k_1$ and $-1/k_2$ is a necessary condition for absolute stability. Encircling the circle from $-1/k_1$ to $-1/k_2$ is a sufficient condition. Hence, if the Nyquist plot touches the circle, no conclusions about the absolute stability can be made

Self-sustaining oscillations are called **limit cycles**. Such cycles occur if $L(j\omega) = -1/N(A)$ where $N(A)$ can be imagined as an amplitude-dependent gain within a specific range. ω is the frequency and A the amplitude of the limit cycle. The number of intersections of $L(j\omega)$ and $N(A)$ equal the number of limit cycles:



If $-1/N(A)$ lies in an unstable region of the Nyquist plot, the amplitude of the oscillations increases. In a stable region, the amplitude decreases. The amplitude continues to increase / decrease until it reaches a limit cycle. There, a change in the amplitude immediately causes it to react back to its original value

Control Design Limitations:

- Non-minimum-phase zeros limit the crossover frequency (closed loop bandwidth)
- Open Loop unstable poles require the crossover frequency to be higher
- It is possible to have small sensitivity at all frequency ranges. Open loop unstable poles firther increase the sensitivity

Damping

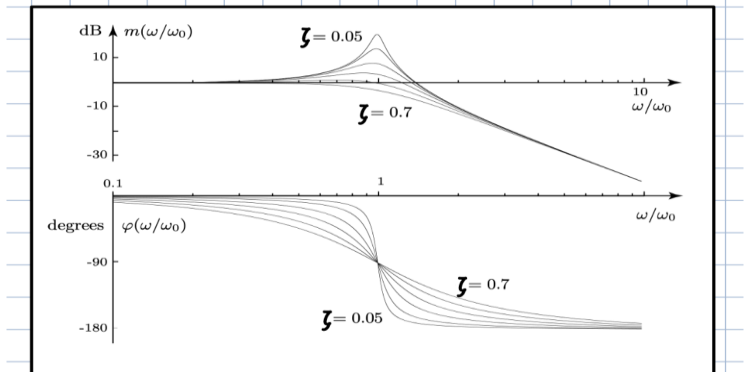


Figure 8.3. Bode diagram of the second-order system (8.15) for six different damping ratios $\zeta = [0.05, 0.1, 0.2, 0.33, 0.5, 0.7]$.

13.2 Describing Functions

Frequency Response of a static nonlinearity (example)
Suppose you apply sinusoidal input to a static nonlinearity:

$$u(t) = Asin(\omega t)$$

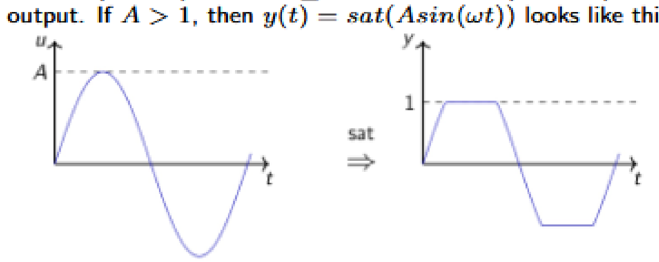
The output will be of the form:

$$y(t) = f(Asin(\omega t))$$

All we can say is that the output y will be a periodic signal with the same frequency as the input. Take for example the **saturation nonlinearity**:

$$sat(u) = \begin{cases} 1 \ \ if \ \ u \geq 1 \\ u \ \ if \ \ -1 < u < 1 \\ -1 \ \ if \ \ u \leq 1 \end{cases}$$

If the input amplitude $A \leq 1$, then the output is equal to the output. If $A > 1$, then $y(t) = sat(Asin(\omega t))$ looks like this:



The output of the non-linearity can be approximated by its first harmonic:
Odd non-linearity: $y(t) \approx b_1 sin(\omega t)$
Even non-linearity: $y(t) \approx a_1 cos(\omega t)$
The ratio of a_1 or b_1 to A is called the **describing function**:

$$N_{odd}(A) = \frac{b_1}{A} = \frac{1}{\pi A}\int_{-\pi}^{\pi} y(t)sin(n\omega t)d(\omega t)$$

$$N_{even}(A) = \frac{a_1}{A} = \frac{1}{\pi A}\int_{-\pi}^{\pi} y(t)cos(n\omega t)d(\omega t)$$

If the input has both odd and even components a more general form of the describing function is given by.

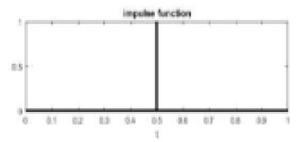
$$N(A,\omega) = \frac{c_1(A,\omega)}{A}e^{j\phi_1(A,\omega)}$$

The new loop transfer function utilizing the describing function is approximated as:

$$L'(A,s) \approx N(A)L(s)$$

3.2 Impulse Response

For an input of an Impulse Function of the form:



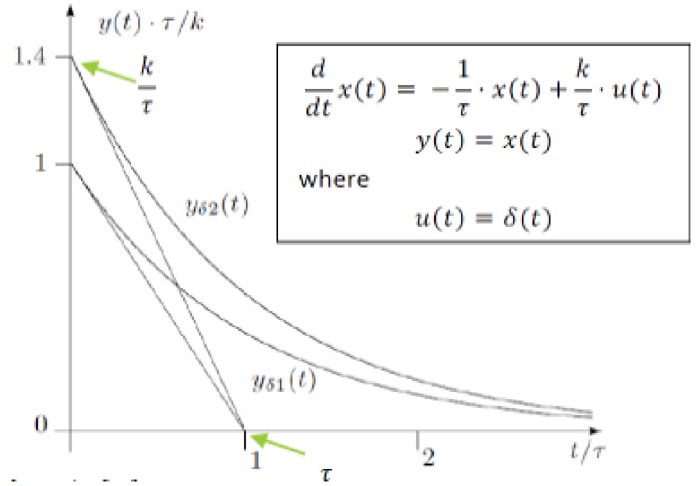
with the input

$$u(t) = \delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

We receive the general solution:

$$\boxed{y_\delta(t) = e^{\frac{-t}{\tau}}\cdot\left(x_0 + \frac{k}{\tau}\right)}$$

with time constant τ and coefficient k. Graphically it is displayed as:



Time Delay Phase

Correct. In the lecture it was shown that the phase margin of the system affected by a time delay is $\phi_{m,T} = \phi_{m,0} - \omega_c T$, where $\phi_{m,0}$ is the phase margin of the system without the time delay, ω_c is the crossover frequency. When increasing T, the phase margin decreases and following this the achievable bandwidth decreases.

Question 38 *Mark all correct statements. (2 Points)*

Mark all of the following statements about control limitations that are true.

- ■ The dominant right-half-plane pole puts a lower limit on the system's bandwidth.
- ■ The sensitivity function, $S(s)$, plus the complementary sensitivity function, $T(s)$, must equal one for all s.
- ■ A plant's right-half-plane zero limits the bandwidth of the closed-loop system.
- ■ When a plant has a right-half-plane zero z, the sensitivity function at z, $S(z)$, must equal 1.

System responses - 1.order

First order system $y = x$

$$\dot{x} = -\frac{1}{\tau}\cdot x + \frac{k}{\tau}\cdot u,$$

time constant τ and gain k

$$G(s) = \frac{k}{\tau s + 1}$$

Impulse response

$$u(t) = \delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

TF +

General solution:

$$y_\delta = e^{\frac{-t}{\tau}}\left(x_0 + \frac{k}{\tau}\right) \quad y(0) = x_0 + \frac{k}{\tau}; \quad \dot{y}(0)\cdot\tau = -y(0)$$

→ τ is the time it would take to reach 0 with the initial $\dot{y}(0)$

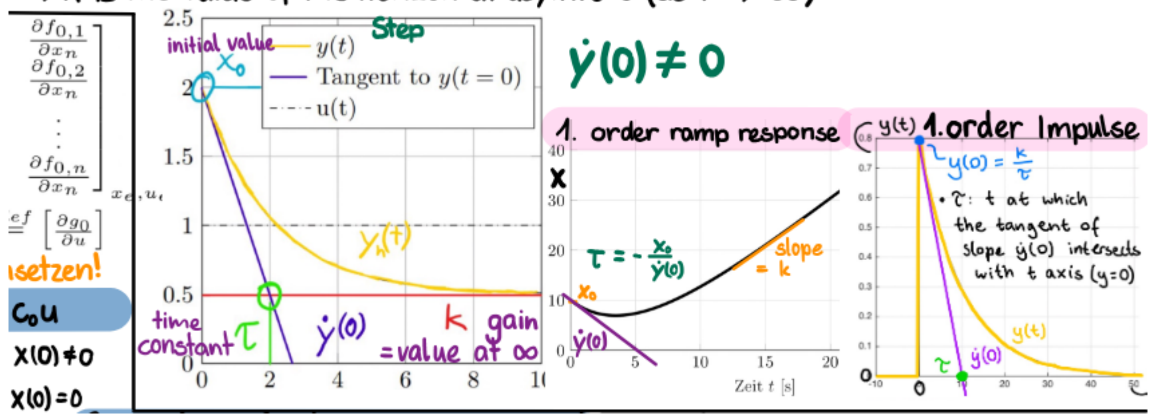
Step response

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{General solution: } y = e^{\frac{-t}{\tau}}\cdot x_0 + k\left(1 - e^{\frac{-t}{\tau}}\right)$$

$$y(0) = x_0; \quad \dot{y}(0)\cdot\tau = k - y(0); \quad k = \lim_{t\to\infty} y(t)$$

→ τ is the time it would take to reach k with the initial $\dot{y}(0)$

→ k is the value of the horizontal asymptote (as $t \to \infty$)



The desired control objectives are,

- no steady-state error to step references,
- tracking error to sinusoidal inputs of frequencies up to 1 rad/s should not exceed 10% in magnitude,
- (noise) frequencies higher than 50 rad/s should be suppressed at least by a factor 10,
- desired phase margin of at least 45°, and,
- a desired closed-loop bandwidth of approximately 10 rad/s.

the controller must consist of an integrator such that the steady-state error is zero.

Through the second requirement, the following is required:

$$|S(j\omega)| < 0.1, \quad \text{when } \omega < 1 \text{ rad/s}$$
$$\frac{1}{|1 + L(j\omega)|} < 0.1$$
$$10 < |1 + L(j\omega)| \approx |L(j\omega)| \text{ since} |L(j\omega)| >> 1 \text{ when } \omega < 1 \text{ rad/s}$$

Since the requirement must hold true up to 1 rad/s:

$$10 < |L(j)| = \left|\frac{10}{j+10}\right| \cdot |C(s)|$$

This is only possible with options C and D.

By the third requirement, noise should be reduced by a factor of 10 for $\omega > 50$ rad/s. Thus:

$$|T(50j)| < \frac{1}{10}|P(50j)|$$

Only option C, $C(s) = \frac{10}{s}$ satisfies this requirement.

By the third requirement, noise should be reduced by a factor of 10 for $\omega > 50$ rad/s. Thus:

$$|T(50j)| < \frac{1}{10}|P(50j)|$$

Only option C, $C(s) = \frac{10}{s}$ satisfies this requirement.

Then ensure that option C satisfies the remaining control objectives.
To check requirement 4, either draw a bode plot approximation, or:

$$|L(j\omega_{gc})| = \left|\frac{10}{j\omega_{gc}} \cdot \frac{10}{j\omega_{gc} + 10}\right| = 1$$
$$100 = \sqrt{\omega_{gc}^2 + 100\omega_{gc}^2}$$
$$100 \approx 10\omega_{gc} \implies \omega_{gc} \approx 10 \text{ rad/s}$$

The phase at the crossover frequency is:

$$\angle L(j\omega_{gc}) = 0 - (90° + 45°) = -135°$$

Hence, the phase margin is approximately 45°, and thus requirement 4 is met.

Similarly, the bandwidth can be approximated by the gain crossover frequency, and thus is around 10 rad/s. Hence, requirement 5 is also met (You can also verify this formally by determining ω_b such that $|T(j\omega_b)| = \frac{1}{\sqrt{2}}$), and the solution is $C(s) = \frac{10}{s}$.